Section 3.3

The Product and Quotient Rules for Differentiation

(1) The Product Rule

(2) The Quotient Rule

Differentiating a Product of Functions

Suppose that $f(x)$ and $g(x)$ are two differentiable functions.
Their product fg is defined by $fg(x) = f(x)g(x)$. **What is $(fg)'(x)$?**

$$\frac{d}{dx}(fg(x)) = \boxed{(fg)'(x)} = \lim_{h\to 0} \frac{\overbrace{f(x+h)g(x+h)}^{fg(x+h)} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h)\overbrace{-f(x+h)g(x) + f(x+h)g(x)}^{=0} - f(x)g(x)}{h}$$

$$= \overbrace{\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}_{=f(x),\ \textbf{by continuity of } f} \left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right)}^{\textbf{By Product Law of Limits}} + \overbrace{g(x)\left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right)}^{\textbf{By Constant Multiple Law of Limits}}$$

$$= \boxed{f(x)g'(x) + g(x)f'(x).}$$

(Note that, f and g are continuous because they are differentiable.)

The Product Rule

To summarize, if $f(x)$ and $g(x)$ are differentiable at $x = a$, then we have just proved that...

(I) The product $fg(x)$ is differentiable at $x = a$.

(II) $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$, or equivalently

$$\frac{d}{dx}\left(fg(x)\right)\Big|_{x=a} = \left(f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}\right)\Big|_{x=a}$$

(These last two equations say the same thing in Lagrange and Leibniz notation respectively.)

The Product Rule

If $f(x)$ and $g(x)$ are differentiable at $x = a$, then $(fg)(x)$ is differentiable at $x = a$ and

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x).$$

Example I:

(I) $\dfrac{d}{dx}\left((4x-3)(3x+5)\right) = 4(3x+5) + 3(4x-3) = 24x + 11$

(II) $\dfrac{d}{dx}\left(\sqrt{x}(x^2+1)\right) = \dfrac{5x^2+1}{2\sqrt{x}}$

(III) $\dfrac{d}{dx}\left(xe^x\right) = (x+1)e^x$

Differentiating a Quotient of Functions

$$\frac{d}{dx}\left(\frac{f}{g}(x)\right) = \left(\frac{f}{g}\right)'(x) = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{g(x)f(x+h) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

$$= \frac{g(x)\left(\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}\right) - f(x)\left(\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}\right)}{g(x)\left(\lim\limits_{h\to 0} g(x+h)\right)}$$

If $f(x)$ and $g(x)$ are differentiable at $x = a$ and $g(a) \neq 0$, then the quotient $\frac{f}{g}(x)$ is differentiable at $x = a$.

The Quotient Rule

If $f(x)$ and $g(x)$ are differentiable at $x = a$ and $g(a) \neq 0$, then $\left(\frac{f}{g}\right)(x)$ is differentiable at $x = a$ and

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Example II:

(I) $\frac{d}{dx}\left(\frac{x^2+1}{\sqrt{x}}\right) = \frac{3x^2-1}{2\sqrt{x^3}}$

(II) $\frac{d}{dx}\left(\frac{e^x}{1+e^x}\right) = \frac{e^x}{(1+e^x)^2}$

Example III, Product Rule

Suppose that

$$f(1) = -2 \qquad g(1) = 4$$
$$f'(1) = -3 \qquad g'(1) = 1$$

(I) **Find $h'(1)$ if $h(x) = 5f(x) - 4g(x)$.**

$$h'(x) = 5f'(x) - 4g'(x)$$
$$h'(1) = 5f'(1) - 4g'(1)$$
$$= 5(-3) - 4(1) = \boxed{-19.}$$

(II) **Find $h'(1)$ if $h(x) = f(x)g(x)$.**

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$
$$h'(1) = f(1)g'(1) + f'(1)g(1)$$
$$= (-2)(1) + (-3)(4) = \boxed{-14.}$$

Example IV, Quotient Rule

Suppose that

$$f(1) = -2 \qquad g(1) = 4$$
$$f'(1) = -3 \qquad g'(1) = 1$$

(I) **Find $h'(1)$ if $h(x) = f(x)/g(x)$.**

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$h'(1) = \frac{(4)(-3) - (-2)(1)}{4^2} = \boxed{\frac{-5}{8}}$$

(II) **Find $h'(1)$ if $h(x) = \frac{1+g(x)}{1-f(x)}$.**

$$h'(x) = \frac{(1 - f(x))(g'(x)) - (1 + g(x))(-f'(x))}{(1 - f(x))^2}$$

$$h'(1) = \frac{(1 - (-2))(1) - (1 + 4)(-(-3))}{(1 - (-2))^2} = \frac{3 - 15}{3^2} = \boxed{-\frac{4}{3}}$$